3D FEM Quasimodal Analysis of the Haroche QED Cavity

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***Abstract*—This paper presents the numerical computation of quasimodes (complex frequencies) of 3D open structures using the Finite Element Method (FEM) combined with Perfectly Matched Layers (PMLs) in order to truncate the infinite domain. The PMLs provide a suitable non-Hermitian extension of the scattering operator associated to the problem and unveil some quasimodes (leaky modes) by rotating the continuous spectrum in the complex plane. The unveiling of the modes depends on the parameters of the PML but the modes themselves are independent of these PML parameters and inhere in the structure. The FEM formulation leads to non-Hermitian matrices with complex eigenvalues that can be numerically computed. The PMLs are presented in the framework of transformation optics as a complex-valued change of coordinates. This model is applied to the Haroche QED cavity (used for the researches that have led to the recent 2012 Nobel Prize of Physics).**

***Index Terms*—Finite element methods, Eigenvalues and eigenfunctions, Cavity resonators.**

I. Resonances of open resonators using perfectly matched layers

The major difficulty in the treatment of open problems in a numerical scheme based on a finite computational window is to deal with infinity issues. Since their introduction by Bérenger in [1] for the time dependent Maxwell's equations, Perfectly Matched Layers (PMLs) have become a widely used technique in computational physics. The idea is to enclose the area of interest by surrounding layers which are absorbing and perfectly reflectionless. These absorbing boundary conditions can be understood in the global framework of transformation optics ([2]). The principle of the technique is to perform a geometrical transformation (here a complex stretch of coordinates), leading to equivalent material properties ([3]). The spectral problem we are dealing with consists in finding the solutions of source free Maxwell's equations, *i.e.* finding complex eigenvalues $\Lambda_n = (\omega_n/c)^2$ and non zero eigenvectors $\vec{E}_n$ such that :

$$\mathcal{M}(\vec{E}_n) := \vec{\nabla} \times \left(\underline{\underline{\mu}}^{-1} \cdot \vec{\nabla} \times \vec{E}_n \right) = \Lambda_n \, \underline{\underline{\varepsilon}} \cdot \vec{E}_n. \qquad (1)$$

where $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\mu}}$ are the relative dielectric permittivity and magnetic permeability tensors describing the electromagnetic properties of the system (cavity+external world).

For Hermitian open problems, the generalized spectrum of Maxwell's operator $\mathcal{M}$ is real and composed of two parts: the discrete spectrum with trapped modes exponentially decreasing at infinity, and the continuous spectrum with radiation modes oscillating at infinity. In addition, another type of solution is present and very useful to characterize the spectral properties of unbounded structures: the so-called *leaky modes* (also termed quasimodes). These eigenmodes with complex associated frequency are an intrinsic feature of open structures. PMLs have proven to be a very convenient tool to compute leaky modes in different configurations ([4], [5]). Indeed they mimic efficiently the infinite space provided a suitable choice of their parameters. The introduction of infinite PMLs rotates the continuous spectrum in the complex plane (since the operator involved in the problem is now a non self-adjoint extension of the original self-adjoint operator). The effect is not only to turn the continuous spectrum into complex values but it also unveils the leaky modes is the region swept by the rotation of this essential spectrum ([6]). It is important to note that leaky modes do not depend on the choice of a particular complex stretching : adding the PMLs is only a way to discover them. Finally, in order to apply the FEM, the PMLs have to be truncated at finite distance which results in an operator having only point spectrum with approximate radiation modes (also termed as PML modes or Bérenger modes) due to the discretization of the continuous spectrum by finite PMLs ([7]). In the sequel, eigenvalues are denoted $\omega_n = \omega'_n + \mathrm{i}\omega''_n$. The real part is the resonant angular frequency $\omega'_n = 2\pi f_n$ and the imaginary part is the damping coefficient, which is related to the lifetime τ_n of the photon in the cavity by $\omega''_n = 2\pi/\tau_n$. The quality factor associated to a resonance is defined by $Q_n = \omega'_n/(2\omega''_n)$.

II. Numerical results

In this section we study the open QED cavity described in [8] by searching for its eigenmodes and complex eigenfrequency, using a Finite Element Method (FEM). This classical electrodynamic approach allow us to derive a number of features observed experimentally by Haroche and coworkers ([9]). A detailed description of the cavity can be found in [8]. It is composed of two mirrors of diameter $D = 50$ mm facing each other. The distance between their apexes is $L = 27.57$ mm, and their surface is toroidal with radii of curvature $r = 39.4$ mm in the Oxz plane and $R = 40.6$ mm in the Oyz plane. The mirrors are coated with a thick layer of

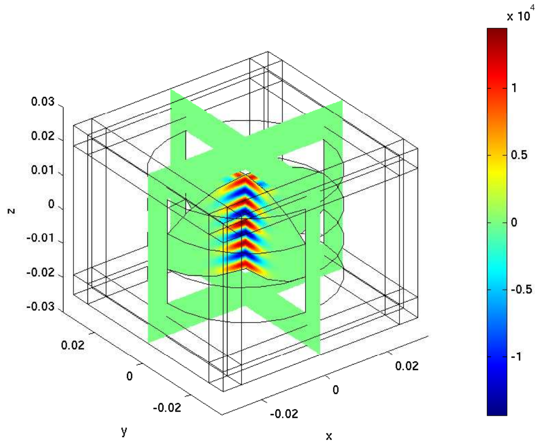

Figure 1. Field maps of a quasimode.

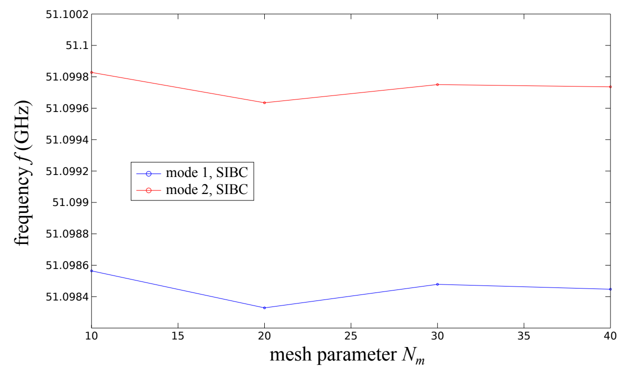

Figure 2. Resonance frequencies of the two longest life modes $vs.$ mesh parameter.

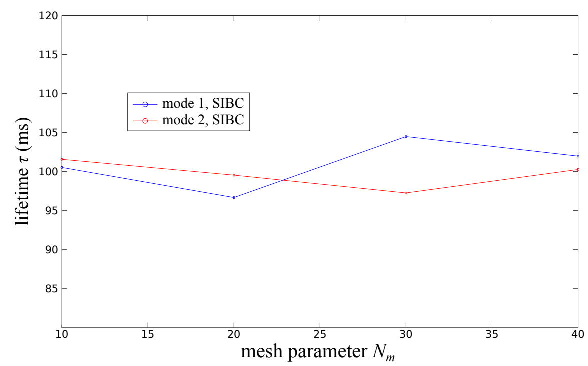

Figure 3. Lifetimes of the two longest life modes $vs.$ mesh parameter.

superconducting niobium. We take advantage of the symmetries of the problem and thus model only one eighth of the cavity to save memory and computation time. By setting a well chosen combination of Neumann and Dirichlet boundary conditions on the cutting planes, we can select the modes with desired symmetries. The eigenproblem defined by Eq.(1) is then solved by the FEM, using cartesian PMLs terminated by homogeneous Neumann boundary conditions to truncate the infinite space. The computational cell is meshed using 2[nd] order edge elements, with a maximum size of an element set to $\frac{\lambda^{\rm r}}{|\Re\mathfrak{e}\,\varepsilon|N}$, where $\lambda^{\rm r} = 3.68$ cm is the approximate resonant wavelength of the cavity, and N is an integer ($N = 7$ for the domain inside of the cavity, $N = 7$ for the domain outside the cavity, $N = 5$ for the PMLs and $N = N_{\rm m}$ for the mirror surfaces). The final algebraic system is solved using a direct solver (PARDISO). In order to account for losses, absorption is considered through a Surface Impedance Boundary Condition (SIBC) with $Z_{\rm s} = X_{\rm s} + {\rm i}Y_{\rm s}$ on the boundaries of both mirrors. London penetration depth for niobium $L_{\rm L}$ (independent of the frequency) is set to a typical value of 0.1 µm and the imaginary (inductive) part of the impedance can be approximated by $Y_{\rm s} = \omega\mu_0 L_{\rm L} = 6.4$ µΩ. As for the real (resistive) part of the impedance, it is extremely difficult to measure and greatly depends on numerous experimental conditions. Therefore, our only option is to estimate this parameter to obtain lifetimes of the same order of magnitude as measured in [8]. Finally, we adjusted the length of the cavity to find resonant frequencies close to those measured in [8]. With the value of L=27.562 mm (instead of 27.57 mm in [8]), the system exhibits two resonant frequencies $f_1 = 51.0984$ GHz and $f_2 = 51.0997$ GHz ($\tau = f_1 - f2 \approx 1.29$ MHz). The convergence as function of mesh refinement is reached (see Fig. 2) for $N_m = 40$. With $X_{\rm s} = 1$ µΩ, we obtain lifetimes of $\sim$100 ms for both modes (see Fig. 3) $i.e.$ $Q_n \sim 2.5\,10^9$, which corresponds to the average lifetime found in [8] for two cavities (supposed to be identical): 112 ± 4 ms (LF mode) and 87 ± 10 ms (HF mode) for M_1, 74 ± 6 ms (LF mode) and 130 ± 4 ms (HF mode) for M_2. This value of $X_{\rm s}$ should be seen as an upper bound of the resistive phenomenon. It includes all other loss processes than radiation loss: roughness of the mirrors, superconductor imperfections… but the fact that the discrepancy between lifetimes corresponding to the two cavity modes is greater experimentally than numerically tends to indicate a residual tilt between the two mirrors.

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